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Dielectric properties of an ultra-cold weakly magnetized charged Bose gas

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Abstract. With a new asymptotic expansion for the specific Kummer function that appears in the response theory of low-temperature quantum plasmas in an external magnetic field, the dielectric properties of the charged Bose gas, namely its collective and transverse modes and screening properties, are evaluated in the weak-field limit.

The interacting Bose gas is a complex and still incompletely solved problem in many-body physics, which may serve as a model for several real physical systems from superconductors [1] and liquid He^4 superfluid [2] to the interiors of exotic astrophysical objects such as white dwarfs, neutron stars and possibly supernovas [3]. One of the most studied examples of the interacting Bose gas is the weakly interacting, charged Bose gas (CBG), first investigated by Foldy [4]. Although much simpler than many of the interacting Bose systems, the CBG continues to be a subject of great interest, since its behaviour is expected to be similar to the more complicated systems possessing long-range interactions [5]. Recently, it has attracted much interest because of its role in the bipolaron theory of high-temperature superconductivity [2, 6].

In contrast to the work of others [7, 8], who have studied the system in the high-temperature classical Boltzmann region, this article aims to present both the longitudinal and transverse dielectric response properties of the CBG in a weak, homogeneous, external magnetic field at $T = 0$ K, when the system is in a total quantum state. Previously, Hore and Frankel (HF) [9] made an attempt to solve this difficult fundamental problem, but their study, which was primarily concerned with the longitudinal properties of the system, failed to liberate the appropriate physics for two reasons. The first was that they evaluated the conductivity tensor via the self-consistent random-phase-approximation (RPA) method developed by Harris [10]. The problem with this approach was that it not only omitted a term necessary for the study of the transverse properties of the system, but it rendered a very awkward form for the tensor. A superior approach was to adapt the Harris method to evaluate the more elegant polarization tensor following Witte *et al* [11] and then to use this tensor's symmetry properties to obtain simplified forms for the various response functions of the system as in [12].

The second, more important, reason why HF failed to liberate such longitudinal properties as the dispersion relation for the collective modes and the static screening potential was that they were unable to develop appropriate asymptotic expansions in the weak magnetic field limit for the particular confluent hypergeometric function or Kummer

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function that appears in the response theory of the magnetized CBG at $T = 0$ K. This Kummer function, which is related to the incomplete gamma function, appears in the response theory of the magnetized CBGs more illustrious Fermi counterpart, the magnetized degenerate electron gas (DEG) and also in the response theory of magnetized anyon gases [13]. Thus, a weak-field asymptotic expansion for this Kummer function is not only required to evaluate the dielectric properties of the CBG at $T = 0$ K, but will also be useful in evaluating the properties of the magnetized DEG. In addition, since the asymptotic expansion is dependent upon the ratio of the cyclotron frequency to the plasma frequency being less than unity, the physical properties presented here are also valid for high densities and strong magnetic fields. For strong magnetic fields at ultra-cold temperatures the finite temperature distribution function of a Bose gas can be replaced by its $T = 0$ K form [14]. Thus, the results presented here should be valid for high-density systems not necessarily restricted to $T = 0$ K.

The longitudinal dielectric response function for a CBG in an external homogeneous magnetic field B for arbitrary temperature [12] is

$$\epsilon_{\mathrm{L}}(\boldsymbol{q},\omega) = 1 + \frac{2m\omega_{\mathrm{c}}e^2}{\hbar q^2 V^{1/3}} \mathrm{e}^{-z_*} \sum_{n,n',p_z} F(n,p_z)\alpha_{n,n'}(z_*) \times (D(E_{n',p_z-\hbar q_z}, E_{n,p_z}) - D(E_{n,p_z}, E_{n',p_z+\hbar q_z})) \tag{1}$$

where $F(n, p_z)$ is the distribution function for the bosons, $D(x, y) = (\hbar\omega + x - y + \mathrm{i}\eta)^{-1}$, $z_* = q_\perp^2/2\beta^2$ ($\beta^2 = eB/\hbar c$) and the Landau energy level, E_{n,p_z}, in terms of the cyclotron frequency $\omega_{\mathrm{c}}(eB/mc)$ is

$$E_{n,p_z} = p_z^2/2m + (n + 1/2)\hbar\omega_{\mathrm{c}}. \tag{2}$$

In equation (1) for $n \geqslant n'$, $\alpha_{n,n'}(z) = \gamma_{n,n'}(z) L_{n'}^{n-n'}(z)^2$ with $\gamma_{n,n'}(z)$ equal to $n'!z^{n-n'}/n!$ and $L_{n'}^{n}(z)$ representing a Laguerre polynomial, while n and n' are interchanged when $n < n'$. This interchanging of n and n' results from the symmetry properties of the polarization tensor. It is implicitly understood that the limit $\eta \to 0^+$ is to be taken here. The $\mathrm{i}\eta$ appears in accordance with the standard Landau procedure [15] of replacing ω by $\omega + \mathrm{i}\eta$, but since there is no singularity in the denominators at $T = 0$ K, it can be discarded. Hence, all modes in the magnetized CBG are zero-damped at $T = 0$ K.

For photon propagation parallel to the magnetic field, i.e. $q = q_z$, the electromagnetic modes are no longer conveniently described in terms of linear polarization, but in terms of circularly polarized waves [16]. For this situation the dielectric tensor becomes diagonal with the components given by

$$\epsilon_{\mathrm{r}}(q,\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} + \frac{2e^2\omega_{\mathrm{c}}^2}{\omega^2 V^{1/3}} \sum_{n,p_z} F(n,p_z)((n+1) \times D(E_{n+1,p_z-\hbar q}, E_{n,p_z}) - nD(E_{n,p_z}, E_{n-1,p_z+\hbar q})) \tag{3}$$

$$\epsilon_{\mathrm{l}}(q,\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} + \frac{2e^2\omega_{\mathrm{c}}^2}{\omega^2 V^{1/3}} \sum_{n,p_z} F(n,p_z)(nD(E_{n-1,p_z-\hbar q}, E_{n,p_z}) - (n+1)D(E_{n,p_z}, E_{n+1,p_z+\hbar q})) \tag{4}$$

and

$$\epsilon(q,\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} + \frac{2e^2\omega_{\mathrm{c}}}{\omega^2 V^{1/3}\hbar m} \sum_{n,p_z} F(n,p_z)((p_z - \hbar q/2)^2 D(E_{n,p_z-\hbar q}, E_{n,p_z}) - (p_z + \hbar q/2)^2 D(E_{n,p_z}, E_{n,p_z+\hbar q})). \tag{5}$$

Equations (3) and (4) correspond to the response functions for left- and right-circularly polarized transverse modes while equation (5) is the longitudinal dielectric response function with $q_\perp = 0$. Due to a transcription error equations (3) and (4) appear in reverse order in [12].

For the case of photon propagation perpendicular to the magnetic field, i.e. $q_z = 0$ or $q_\perp = q$ with the photons chosen to propagate in the x-direction, the transverse response functions were found to be

$$\epsilon_1(q,\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} + \frac{e^2\omega_\mathrm{c}^2\mathrm{e}^{-z_*}}{z_*\omega^2 V^{1/3}\hbar} \sum_{p_z,n,n'} (n-n')^2 \alpha_{n,n'}(z_*) \times F(n,p_z)(D(n'\hbar\omega_\mathrm{c}, n\hbar\omega_\mathrm{c}) - D(n\hbar\omega_\mathrm{c}, n'\hbar\omega_\mathrm{c})) \tag{6}$$

$$\epsilon_2(q,\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} + \frac{e^2\omega_\mathrm{c}^2\mathrm{e}^{-z_*}}{z_*\omega^2 V^{1/3}} \sum_{p_z,n,n'} \beta_{n,n'}(z_*) \times F(n,p_z)(D(n'\hbar\omega_\mathrm{c}, n\hbar\omega_\mathrm{c}) - D(n\hbar\omega_\mathrm{c}, n'\hbar\omega_\mathrm{c})) \tag{7}$$

and

$$\epsilon_3(q,\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} + \frac{2e^2\omega_\mathrm{c}^2\mathrm{e}^{-z_*}}{m^2\omega^2 V^{1/3}} \sum_{p_z,n,n'} p_z^2 \alpha_{n,n'}(z_*) \times F(n,p_z)(D(n'\hbar\omega_\mathrm{c}, n\hbar\omega_\mathrm{c}) - D(n\hbar\omega_\mathrm{c}, n'\hbar\omega_\mathrm{c})) \tag{8}$$

where the plasma frequency is given by $\omega_\mathrm{p}^2 = 4\pi e^2 N/mV$ and N is the total number of particles in the system. In equation (7), $\beta_{n,n'}(z) = \gamma_{n,n'}(z)(nL_{n'}^{n-n'-1}(z) - zL_{n'}^{n-n'+1}(z))^2$ for $n \geqslant n'$ while n and n' are interchanged when $n < n'$. Equations (6)–(8) represent the diagonal components of the dielectric tensor. One off-diagonal transverse response function also exists which is given by

$$\epsilon_x(q,\omega) = \frac{e^2\omega_\mathrm{c}^2\mathrm{e}^{-z_*}}{z_*\omega^2 V^{1/3}} \sum_{p_z,n,n'} \kappa_{n,n'}(z_*) F(n,p_z)(D(n\hbar\omega_\mathrm{c}, n'\hbar\omega_\mathrm{c}) - D(n'\hbar\omega_\mathrm{c}, n\hbar\omega_\mathrm{c})). \tag{9}$$

In equation (9) $\kappa_{n,n'}(z) = \gamma_{n,n'}(z)(nL_{n'}^{n-n'-1}(z)(zL_{n'}^{n-n'}(z) - nL_{n'}^{n-n'-1}(z)) - z^2L_{n'}^{n-n'}(z)L_{n'-1}^{n-n'+1}(z))$ for $n \geqslant n$ while n and n' are interchanged when $n < n'$.

To study the magnetized CBG in its total quantum region, we now introduce the $T = 0$ K distribution function into the various response functions given above. At $T = 0$ K, all the bosons are in the lowest energy level and hence, $F(n, p_z) = 2\pi N\delta_{n,0}\delta_{p_z,0}/\beta^2 V^{2/3}$. Introducing this distribution function into the longitudinal dielectric response function given by equation (1) yields

$$\epsilon_\mathrm{L}(\boldsymbol{q},\omega) = 1 + \frac{m\omega_\mathrm{p}^2}{q^2}[a_+^{-1}\Phi(1, 1+a_+/b; -z_*) - a_-^{-1}\Phi(1, 1+a_-/b; -z_*)] \tag{10}$$

where $a_\pm = \hbar\omega \pm \hbar^2 q_z^2/2m$, $b = \hbar\omega_\mathrm{c}$ and $\Phi(1, 1+a/b; z)$ is a confluent hypergeometric or Kummer function, which first appeared in the study by Witte *et al* [11] of the magnetized relativistic CBG. This function will be referred to as the Bose–Kummer function, but as mentioned earlier, it appears in the response theory of the magnetized DEG and is also related to the incomplete gamma function. It should also be noted that due to charge conjugation ω_c should be replaced by $|\omega_\mathrm{c}|$ for negatively charged bosons in all the response functions given above.

By inserting the $T = 0$ K distribution function into the response functions for right- and left-circularly polarized modes, i.e. equations (3) and (4), one obtains

$$\epsilon_{\rm r}(q,\omega) = 1 - \frac{\omega_{\rm p}^2}{\omega^2} + \frac{\omega_{\rm p}^2}{\omega^2}\left(\frac{\omega_{\rm c}}{\omega + \omega_{\rm c} + \hbar q^2/2m}\right) \tag{11}$$

and

$$\epsilon_{\rm l}(q,\omega) = 1 - \frac{\omega_{\rm p}^2}{\omega^2} - \frac{\omega_{\rm p}^2}{\omega^2}\left(\frac{\omega_{\rm c}}{\omega - \omega_{\rm c} - \hbar q^2/2m}\right) \tag{12}$$

whilst the longitudinal dielectric function given by equation (5) becomes

$$\epsilon(q,\omega) = 1 - \frac{\omega_{\rm p}^2}{\omega^2 - \hbar^2 q^4/4m^2}. \tag{13}$$

Equation (13) represents the one-dimensional version of the field-free longitudinal dielectric response function obtained by HF [5, 9]. The dispersion relation for longitudinal/plasmon modes propagating parallel to the magnetic field can be obtained by setting this equation equal to zero. As we shall see shortly, the same result will be obtained by putting $q_\perp = 0$ in the general longitudinal dielectric response function given by equation (10). For $|\omega \pm \omega_{\rm c}| \gg \hbar q^2/2m$, equations (11) and (12) reduce to those for a classical electron plasma in the cold-plasma limit [16].

Although the Bose–Kummer function does not appear when studying the propagation of electromagnetic modes parallel to the magnetic field, it does appear when the $T = 0$ K distribution function is introduced into the transverse response functions for analysing the propagation of electromagnetic modes perpendicular to the magnetic field. Then one finds

$$\epsilon_1(q,\omega) = 1 + \frac{m\omega_{\rm p}^2}{\hbar q^2\omega}[\Phi(1, 1+\nu; -z_*) - \Phi(1, 1-\nu; -z_*)] \tag{14}$$

$$\epsilon_2(q,\omega) = 1 - \frac{2\omega_{\rm p}^2}{\omega^2} + \frac{m\omega_{\rm p}^2}{\hbar q^2\omega}\left[\left(1 + \frac{\hbar q^2}{2m\omega}\right)^2 \Phi(1, 1+\nu; -z_*) - \left(1 - \frac{\hbar q^2}{2m\omega}\right)^2 \Phi(1, 1-\nu; -z_*)\right] \tag{15}$$

$$\epsilon_3(q,\omega) = 1 - \omega_{\rm p}^2/\omega^2 \tag{16}$$

and

$$\epsilon_x(q,\omega) = \frac{m\omega_{\rm p}^2}{\hbar q^2\omega}[(1 + \hbar q^2/2m\omega)\Phi(1, 1+\nu; -z_*) - (1 - \hbar q^2/2m\omega)\Phi(1, 1-\nu; -z_*)] \tag{17}$$

where $\nu = \omega/\omega_{\rm c}$.

It can be seen in the weakly magnetized limit, i.e. $\omega_{\rm c} \to 0$, that not only does z_* become large, but also the parameters of $1 + a_\pm/b$ and ν in the Bose–Kummer function become large. As stated in [17] existing asymptotic expansions did not cover this case adequately, so a novel asymptotic expansion was sought, which could ultimately be used to yield the already known field-free results. First, the Bose–Kummer function was expressed by means of a Kummer transformation as

$$\Phi(1, 1+\alpha; -z) = \alpha e^{-z} S(\alpha, z) = e^{-z}\sum_{k=0}^{\infty} \frac{\alpha z^k}{(k+\alpha)k!} \tag{18}$$

where it is $S(\alpha, z)$ that is related to the incomplete gamma function. Then $S(\alpha, z)$ was re-cast into an integral representation and the method of expanding most of the exponential as described in [18] was used in conjunction with a novel graphical technique to arrive at

$$S(\alpha, x) \sim e^x \sum_{k=0}^{\infty} \Gamma(k+1)(\alpha + x)^{-k-1} c_k(x) \tag{19}$$

where the $c_k(x)$ are new polynomials whose highest order in x is $[k/2]$ or the greatest integer less than $k/2$. In addition to general expressions for the coefficients of the three lowest and three highest orders of the $c_k(x)$, it has been found in [19] that equation (19) is very accurate for $|\alpha + x| > 3$. Here, however, we only need the first six polynomials, which are: $c_0(x) = 1$, $c_1(x) = 0$, $c_2(x) = x/2!$, $c_3(x) = -x/3!$, $c_4(x) = x/4! + x^2/4 \times 2!$ and $c_5(x) = -x/5! - x^2/2 \times 3!$.

If equation (19) is introduced into equation (1), then the longitudinal dielectric-response function becomes

$$\epsilon_{\rm L}(\boldsymbol{q}, \omega) \sim 1 + \frac{m\omega_{\rm p}^2}{\hbar q^2} \sum_{k=0}^{\infty} \frac{\omega_{\rm c}^k c_k(\hbar q_\perp^2/2m\omega_{\rm c}) k!}{(\hbar^2 q^4/4m^2 - \omega^2)^{k+1}} \times [(\hbar q^2/2m - \omega)^{k+1} + (\hbar q^2/2m + \omega)^{k+1}]. \tag{20}$$

The dispersion relation for longitudinal modes or plasmons is found by putting $\epsilon_{\rm L}(\boldsymbol{q}, \omega)$ equal to zero. By retaining only those terms up to $\omega_{\rm c}^2$ and then carrying out a perturbational analysis of the ensuing equation, one obtains

$$\omega^2 = \omega_{\rm p}^2 + \frac{\hbar^2 q^4}{4m^2} + \omega_{\rm c}\left(\frac{3\hbar q_\perp^2}{2m} + \frac{\hbar^3 q^4 q_\perp^2}{2m^3\omega_{\rm p}^2}\right) + \omega_{\rm c}^2\left(\frac{q_\perp^2}{q^2}\left(1 + \frac{2\hbar^2 q^4}{m^2\omega_{\rm p}^2} + \frac{\hbar^4 q^8}{2m^4\omega_{\rm p}^4}\right) + \frac{3}{4}\frac{\hbar^2 q_\perp^4}{m^2\omega_{\rm p}^2}\left(-1 + \frac{4\hbar^2 q^4}{m^2\omega_{\rm p}^2} + \frac{2\hbar^4 q^8}{m^4\omega_{\rm p}^4}\right)\right) + \cdots. \tag{21}$$

In the field-free limit, i.e. $\omega_{\rm c} \to 0$, equation (21) reduces to the result obtained first by Foldy [4] and later by HF [5, 9]. It should be noted that Foldy's definition of $\omega_{\rm p}$ is different because it refers to a depleted ground-state occupation that arises from the Bogoliubov approximation [2] as opposed to the RPA used here. An interesting feature of equation (21) is that the weakly magnetized CBG behaves like a field-free CBG when the modes propagate purely in the magnetic field direction, whereas the magnetic-field effects on plasmons are strongest when $q_z = 0$.

The dispersion relations for circularly polarized modes propagating parallel to the magnetic field are obtained by putting equations (11) and (12) equal to $(qc/\omega)^2$. For the weak magnetic-field case, a perturbational approach for right-circularly polarized modes yields

$$\omega^2 = \omega_{\rm T}^2 + \omega_{\rm c}\omega_{\rm p}^2 f_- + \omega_{\rm c}^2\omega_{\rm p}^2 f_-^2\left(1 - \frac{\omega_{\rm p}^2 f_-}{2\omega_*}\right) + \cdots \tag{22}$$

whilst for left-circularly polarized modes

$$\omega^2 = \omega_{\rm T}^2 + \omega_{\rm c}\omega_{\rm p}^2 f_+ + \omega_{\rm c}^2\omega_{\rm p}^2 f_+^2\left(\frac{\omega_{\rm p}^2 f_+}{2\omega_*} - 1\right) + \cdots \tag{23}$$

where $\omega_{\rm T}^2 = \omega_{\rm p}^2 + q^2c^2$ and $f_\pm = (\omega_{\rm T} \pm \hbar q^2/2m)^{-1}$. Equations (22) and (23) represent perturbations around the dispersion relation for transverse modes for the field-free case of $\omega^2 = \omega_{\rm T}^2$. This dispersion relation also applies to the ordinary mode, which is obtained by setting $\epsilon_3(q,\omega)$ in equation (16) equal to $(qc/\omega)^2$. Thus, the ordinary mode is not affected by the presence of a magnetic field at $T = 0$ K and like the circularly polarized modes, it can only propagate when its frequency is greater than the plasma frequency.

It should also be noted that a low frequency mode can be obtained from the dispersion relation for left-circularly polarized modes, which is

$$\omega = \frac{\hbar q^2}{2m} + \frac{\omega_c q^2 c^2}{\omega_p^2 + q^2 c^2}. \tag{24}$$

The above result is valid for those cyclotron frequencies satisfying the condition, $0 < \omega_c < \omega - \hbar q^2/2m$. This low frequency mode is the analogue of the helicon mode found in a magnetized electron plasma [16] and is of interest because it means that photon propagation can occur parallel to the magnetic field with frequencies significantly lower than both the plasma and cyclotron frequencies.

Equation (11) has a resonance at $\omega = \omega_c + \hbar q^2/2m$ where left-circularly polarized photons are strongly absorbed by the system while for negative values of this frequency equation (12) possesses a resonance. Furthermore, near cut-off frequencies the modes have extremely large phase velocities and dissipation processes in the system become insignificant. Modes with frequencies below the cut-off values do not propagate in a plasma. For a field-free plasma the cut-off frequency of circularly polarized modes occurs at the plasma frequency, which implies that a plasma is not able to support these modes below the plasma frequency. Cut-off frequencies are found by solving

$$\frac{\omega_p^2}{\omega^2} = \frac{\omega \pm \omega_c \pm \hbar q^2/2m}{\omega \pm \hbar q^2/2m} \tag{25}$$

where '+' applies to right-circularly polarized modes while '−' applies to left-circularly polarized modes. By assuming that ω_c is small, and carrying out a perturbational analysis around $\omega = \omega_p$, one finds for left-circularly polarized modes that

$$\omega = \omega_p \left(1 + \frac{\omega_c}{2\kappa_-} + \frac{\omega_c^2}{2\kappa_-^2}\left(\frac{3}{4} - \frac{\omega_p}{2\kappa_-}\right) + \cdots\right) \tag{26}$$

while for right-circularly polarized modes one obtains

$$\omega = \omega_p \left(1 + \frac{\omega_c}{2\kappa_+} + \frac{\omega_c^2}{2\kappa_+^2}\left(\frac{3}{4} - \frac{\omega_p}{2\kappa_+}\right) + \cdots\right) \tag{27}$$

where $\kappa_\pm = \omega_p \pm \hbar q^2/2m$.

The dispersion relation for the extraordinary mode [16], often referred to as a hybrid mode splits because its electric field consists of longitudinal and transverse components, is given by

$$(qc/\omega)^2 = (\epsilon_1\epsilon_2 - \epsilon_x^2)/\epsilon_1. \tag{28}$$

The dispersion relation at $T = 0$ K is determined by introducing equation (19) into equations (14), (15) and (17), which eventually yields

$$\psi\left(1 - \frac{\omega_p^2}{\chi}\right) - \frac{A_1\omega_c\psi}{\chi^3} - \frac{B_1\omega_c^2\psi}{\chi^5} + \frac{C_1\omega_c\omega^2}{\chi^2}\left(1 - \frac{\omega_p^2}{\chi}\right) + \frac{A_1C_1\omega_c^2\omega^2}{\chi^5} \approx \frac{4z_*^2\omega_c^4\omega^2\omega_p^4}{\chi^4} \tag{29}$$

where

$$\chi = \omega^2 - z_*^2\omega_c^2 \qquad \psi = \omega^2 - \omega_p^2 - q^2c^2 \qquad A_1 = z_*\omega_c\omega_p^2(z_*^2\omega_c^2 + 3\omega^2)$$
$$B_1 = \omega_p^2\left(2z_*^6\omega_c^6 + 25\omega^2 z_*^4\omega_c^4 + 20\omega^4 z_*^2\omega_c^2 + \omega^6\right) \qquad \text{and} \qquad C_1 = 2z_*\omega_c\omega_p^2. \tag{30}$$

In the $\omega_c \to 0$ limit, equation (29) possesses two branches. To obtain ω_c corrections for the first branch, one must also assume that $z_*^2\omega_c^2 \ll q^2c^2$ or $q^2 \ll 4m^2c^2/\hbar^2$. Carrying out a perturbational analysis then yields

$$\omega^2 \approx \omega_T^2 + \frac{2z_*\omega_c^2\omega_p^2q^2c^2}{\omega_T^2q^2c^2 - 3z_*\omega_c^2\omega_p^2} + \cdots. \tag{31}$$

Thus, to zeroth order in ω_c, the first branch gives the same dispersion relation as the ordinary mode. The second branch can also be found to first order in ω_c by a perturbational approach, which gives

$$\omega^2 \approx \omega_L^2 + \frac{(3\omega_L^2 + z_*^2\omega_c^2)z_*\omega_c^2}{\omega_p^2(q^2c^2 - z_*^2\omega_c^2) + 2z_*\omega_c^2\omega_L^2} + \cdots \tag{32}$$

where $\omega_L^2 = \omega_p^2 + z_*^2\omega_c^2$, the field-free longitudinal dispersion relation obtained by Foldy [4]. The two branches, thus, exhibit the hybrid nature of the extraordinary mode.

The resonances of this mode occur, known as hybrid resonances, when qc/ω is infinite or when $\epsilon_1(q,\omega) = 0$. However, one does not need to solve this equation, because under resonance conditions, the mode is purely longitudinal. Thus, the transverse response function $\epsilon_1(q,\omega)$ is equivalent to the longitudinal dielectric response function $\epsilon_L(q,\omega)$ with $q_z = 0$ [12]. Hence, equation (21) with $q_z = 0$ represents the condition that the extraordinary mode turns into a purely longitudinal mode.

Cut-off frequencies for the extraordinary mode occur whenever the r.h.s. of equation (28) equals zero. From the preceding material we expect to find that the cut-off frequency for a weakly magnetized CBG will occur near the plasma frequency as in the field-free case, but with an additional magnetic-field correction term. The cut-off frequency for a weakly magnetized CBG at $T = 0$ K is obtained by setting the r.h.s. of equation (28) equal to zero. To first order in ω_c, one obtains

$$(1 - \omega_p^2/\omega^2)(\omega^2 - z_*^2\omega_c^2)^3 - A_1\omega_c - C_1\omega_c(\omega^2 - z_*^2\omega_c^2 - \omega_p^2) = 0. \tag{33}$$

A perturbational analysis near the plasma frequency yields

$$\omega^2 = \omega_p^2 + \frac{z_*\omega_c^2\omega_p^4(3\omega_L^2 + z_*^2\omega_c^2)}{(\omega_p^2 - z_*^2\omega_c^2)^3} + \cdots. \tag{34}$$

Since $\omega_p^2 \gg z_*^2\omega_c^2$, one can put the first-order term in equation (34) equal to $3z_*\omega_c^2$.

Finally, the static screening potential for a test-charge Q immersed in a plasma is given by

$$V(r) = (2\pi)^{-3}\int \mathrm{d}\boldsymbol{q}\, \exp(\mathrm{i}\boldsymbol{q}\cdot\mathbf{r})\frac{4\pi Q}{q^2\epsilon_L(\boldsymbol{q},0)}. \tag{35}$$

To first order in ω_c, equation (20) yields $\epsilon(\boldsymbol{q},0) \sim 1 + A^4/q^4 + Eq_\perp^2/q^8$ where $A^2 = 2m\omega_p/\hbar$ and $E = 8m^3\omega_p^2\omega_c/\hbar^3$. Thus the screening integral becomes

$$V(r) = \frac{Q}{\pi}\int_{-\infty}^{\infty}\mathrm{d}q_z\int_0^{\infty}\mathrm{d}q_\perp\frac{q_\perp(q_\perp^2+q_z^2)^3\exp(\mathrm{i}q_z z)J_0(\rho q_\perp)}{(q_\perp^2+q_z^2)^4 + A^4(q_\perp^2+q_z^2)^2 + Eq_\perp^2}. \tag{36}$$

Now if we put $q_z = q\cos\theta$, then $\mathrm{d}q_\perp\mathrm{d}q_z = q\mathrm{d}q\mathrm{d}\theta$ and the screening integral can be evaluated by employing a combination of analytic techniques, whereupon one obtains

$$V(r) \sim Q\mathrm{e}^{-A\lambda_- r/\sqrt{2}}\left[f(z,r)\cos\left(\frac{A\lambda_+ r}{\sqrt{2}}\right) + g(z,r)\sin\left(\frac{A\lambda_+ r}{\sqrt{2}}\right)\right] + Q\frac{\omega_c(r^2-3z^2)}{\omega_p A^2 r^5}\mathrm{e}^{-\sqrt{2m\omega_c/\hbar}\,r} \tag{37}$$

where $\lambda_{\pm} = 1 \pm \omega_c/4\omega_p$, $f(z,r) = 1/r - (\gamma/\sqrt{2}Ar^2)(\sqrt{2} + 1 + A^2z^2/4 - 3z^2/r^2 - 3\sqrt{2}z^2/Ar^3)$, $g(z,r) = (-\gamma/4r)(1 - \sqrt{2}Az^2/r + 2\sqrt{2}/Ar - 5z^2/r^2 - 6\sqrt{2}z^2/Ar^3)$, and $\gamma = \omega_c/\omega_p$. This potential gradually becomes isotropic for large r, provided $z \not\approx r$. In the limit as $\omega_c \to 0$ this potential also reduces to the field-free result of

$$V(r) = \frac{Q}{r}\exp(-Ar/\sqrt{2})\cos(Ar/\sqrt{2}) \tag{38}$$

which was obtained by HF in [5, 9]. Equation (38) is a peculiar result because irrespective of the charge on the test particle, the potential alternates in sign over rings around the particle, although it is damped rapidly.

To conclude, we have seen that the novel asymptotic expansion given by equation (19) is able to transform the highly anisotropic expressions for the dielectric properties of a weakly magnetized CBG into the forms for the isotropic system with weak field perturbations. The results presented here are valid for high densities and strong magnetic fields provided $|\omega_c/\omega_p| < 1$ and hence, are not necessarily restricted to $T = 0$ K. In the future this powerful asymptotic expansion will be employed in a study dealing with the response of the more important weakly magnetized DEG, in both two and three dimensions.

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